

# Terwilliger Algebras of Some Group Association Schemes

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## Abstract

The Terwilliger algebra plays an important role in the theory of association schemes. The present paper gives the explicit structures of the Terwilliger algebras of the group association schemes of the finite groups  $PSL(2, 7)$ ,  $A_6$ , and  $S_6$ .

## 1 Introduction

Association schemes enable us to study combinatorial problems in a unified way. We refer to [2, 4] for the foundations of association schemes. In a series of papers [7], Terwilliger introduced a new method, the so-called Terwilliger algebra, to investigate the commutative association schemes. Since then there have been many investigations on Terwilliger algebras (*cf.* [6, 5]). It is very important to know the explicit structure of the Terwilliger algebra. The cases of the group association schemes of  $S_5$  and  $A_5$  were studied in [1] along the line of the work [3]. In the present paper we determine the structures of the Terwilliger algebras of the group association schemes of the finite groups  $PSL(2, 7)$ ,  $A_6$ , and  $S_6$ .

## 2 Preliminaries

We begin with the definition of a group association scheme.

**Definition 2.1.** Let  $G$  be a finite group and  $C_0 = \{e\}, C_1, \dots, C_d$  the conjugacy classes of  $G$  where  $e$  is the identity of  $G$ . Define the relations  $R_i (i = 0, 1, \dots, d)$  on  $G$  by

$$(x, y) \in R_i \iff yx^{-1} \in C_i.$$

Then  $\mathfrak{X}(G) = (G, \{R_i\}_{0 \leq i \leq d})$  forms a commutative association scheme of class  $d$  called the *group association scheme of  $G$* .

We associate the matrix  $A_i$  of the relation  $R_i$  as

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$$

and  $A_0, \dots, A_d$  generate the so-called Bose-Mesner algebra  $\mathfrak{A}$ . The intersection numbers  $p_{ij}^k$  of the group association scheme  $\mathfrak{X}(G)$  are given by

$$|\{(x, y) \in C_i \times C_j \mid xy = z, z \in C_k\}|.$$

The algebra  $\mathfrak{A}$  has a second basis  $E_0, \dots, E_d$  of primitive idempotents, and

$$E_i \circ E_j = \frac{1}{|G|} q_{ij}^k E_k,$$

where  $\circ$  denotes Hadamard (entry-wise) multiplication. For each  $i = 0, \dots, d$ , let  $E_i^*$  and  $A_i^*$  be the diagonal matrices of size  $|G| \times |G|$  which are defined as follows.

$$(E_i^*)_{x,x} = \begin{cases} 1, & \text{if } x \in C_i \\ 0, & \text{if } x \notin C_i \end{cases} \quad (x \in G),$$

$$(A_i^*)_{x,x} = |G|(E_i)_{e,x} \quad (x \in G).$$

Then  $E_0^*, \dots, E_d^*$  form a basis for the dual Bose-Mesner algebra  $\mathfrak{A}^*$ . The intersection numbers provide information about the structure of the Terwilliger algebra. We refer the following relations [7].

$$\begin{aligned} E_i^* A_j E_k^* = 0 &\Leftrightarrow p_{ij}^k = 0 \quad (0 \leq i, j, k \leq d), \\ E_i A_j^* E_k = 0 &\Leftrightarrow q_{ij}^k = 0 \quad (0 \leq i, j, k \leq d). \end{aligned}$$

We need to fix the ordering of the conjugacy classes. The following table gives the representatives and the orders of conjugacy classes.

1.  $PSL(2, 7)$

	$C_0$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$
rep.	(1)	(357)(468)	(2354786)	(2465837)	(12)(34)(58)(67)	(1235)(4876)
$ C_i $ :	1	56	24	24	21	42

2.  $A_6$

	$C_0$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$
rep.	(1)	(12)(34)	(123)	(123)(456)	(1234)(56)	(12345)	(12346)
$ C_i $	1	45	40	40	90	72	72

3.  $S_6$

	$C_0$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$
rep.	(1)	(12)	(12)(34)	(12)(34)(56)	(123)	(123)(45)
$ C_i $	1	15	45	15	40	120
	$C_6$	$C_7$	$C_8$	$C_9$	$C_{10}$	
	(123)(456)	(1234)	(1234)(56)	(12345)	(123456)	
	40	90	90	144	120	

Finally we give the definition of the Terwilliger algebra of the group association scheme. We shall denote by  $\mathcal{M}_k$  the ring of  $k \times k$  matrices over the complex number  $\mathbf{C}$ .

**Definition 2.2.** Let  $G$  be a finite group. The Terwilliger algebra  $T(G)$  of the group association scheme  $\mathfrak{X}(G)$  is a sub-algebra of  $\mathcal{M}_{|G|}$  generated by  $\mathfrak{A}$  and  $\mathfrak{A}^*$ .

Since  $T(G)$  is closed under the conjugate-transpose,  $T(G)$  is semi-simple. In the next section, we investigate the Terwilliger algebras of the groups association schemes of  $PSL(2, 7)$ ,  $A_6$  and  $S_6$ .

### 3 Results

In [1], Balmaceda and Oura gave the structures of the Terwilliger algebra of the group association schemes of  $S_5$  and  $A_5$ . Following their method, we determine the Terwilliger algebras for the cases  $PSL(2, 7)$ ,  $A_6$ , and  $S_6$ .

**Theorem 3.1.** *The dimensions of  $T(PSL(2, 7))$ ,  $T(A_6)$ , and  $T(S_6)$  are given as follows.*

$$\begin{aligned} \dim T(PSL(2, 7)) &= 165, \\ \dim T(A_6) &= 336, \\ \dim T(S_6) &= 758. \end{aligned}$$

*Proof.* We compute a set of linearly independent elements among  $E_i^* A_j E_k$  and  $E_i^* A_j E_k \cdot E_k^* A_l E_m^* = E_i^* A_j E_k^* A_l E_m^*$ .  $\square$

We provide matrices below to show how many elements of a basis occur. As these matrices are symmetric, we omit the entries below diagonal.

$$PSL(2, 7) : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 13 & 7 & 7 & 5 & 10 \\ & & 6 & 6 & 3 & 6 \\ & & & 6 & 3 & 6 \\ & & & & 4 & 5 \\ & & & & & 9 \end{pmatrix}$$

$$A_6 : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 9 & 5 & 5 & 9 & 7 & 7 & & & \\ & & 8 & 8 & 9 & 8 & 8 & & & \\ & & & 8 & 9 & 8 & 8 & & & \\ & & & & 16 & 13 & 13 & & & \\ & & & & & 12 & 12 & & & \\ & & & & & & 12 & & & \end{pmatrix}$$

$$S_6 : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 3 & 3 & 4 & 4 & 5 & 3 & 2 & 4 & 2 & 3 \\ & & 6 & 4 & 6 & 9 & 8 & 2 & 6 & 4 & 6 \\ & & & 8 & 8 & 8 & 7 & 4 & 8 & 4 & 8 \\ & & & & 12 & 13 & 13 & 4 & 12 & 6 & 13 \\ & & & & & 19 & 16 & 3 & 13 & 6 & 12 \\ & & & & & & 23 & 3 & 13 & 8 & 16 \\ & & & & & & & 3 & 4 & 3 & 5 \\ & & & & & & & & 12 & 6 & 13 \\ & & & & & & & & & 6 & 9 \\ & & & & & & & & & & 19 \end{pmatrix}$$

We denote by  $Z(T(G))$  the center of the Terwilliger algebra  $T(G)$  of a finite group  $G$ .

**Lemma 3.1.** *The dimensions of  $Z(T(G))$  for  $G = PSL(2, 7)$ ,  $A_6$ ,  $S_6$  are given as follows.*

$$\begin{aligned} \dim Z(T(PSL(2, 7))) &= 7, \\ \dim Z(T(A_6)) &= 10, \\ \dim Z(T(S_6)) &= 14. \end{aligned}$$

*Proof.* The result is obtained by determining a basis for center. We solve a linear equation system  $\{x_i y = y x_i\}$  ranging over all elements  $x_i$  in the basis of  $T(G)$  and  $y = \sum c_j b_j$  where  $b_j$  are the basis elements of  $T(G)$  and  $c_j$  is any scalar.  $\square$

Let  $\{e_i : 1 \leq i \leq s\}$  be a basis of  $Z(T(G))$ . Then we have  $e_i e_j = \sum t_{ij}^k e_k$  and put  $B_i = (t_{ij}^k)$  for  $i = 1, 2, \dots, s$ . Since these matrices mutually commute, they are simultaneously diagonalizable. We shall denote by  $v_1(i), \dots, v_s(i)$  the diagonal entries of the diagonalized matrix of  $B_i$  and define the matrix  $M$  by  $M_{ij} = v_i(j)$ . Then we get the primitive central idempotents  $\varepsilon_1, \dots, \varepsilon_s$  by

$$(\varepsilon_1, \dots, \varepsilon_s) = (e_1, \dots, e_s) M^{-1}.$$

**Theorem 3.2.** *The degrees of the irreducible complex representations afforded by every idempotent are given below.*

$T(PSL(2,7))$	$\varepsilon_i$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$	$\varepsilon_6$	$\varepsilon_7$			
	$\deg \varepsilon_i$	1	2	3	3	5	6	9			
$T(A_6)$	$\varepsilon_i$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$	$\varepsilon_6$	$\varepsilon_7$	$\varepsilon_8$	$\varepsilon_9$	$\varepsilon_{10}$
	$\deg \varepsilon_i$	1	3	3	4	4	6	6	7	8	10
$T(S_6)$	$\varepsilon_i$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$	$\varepsilon_6$	$\varepsilon_7$	$\varepsilon_8$	$\varepsilon_9$	$\varepsilon_{10}$
	$\deg \varepsilon_i$	1	1	1	3	3	4	6	7	8	8
		$\varepsilon_{11}$	$\varepsilon_{12}$	$\varepsilon_{13}$	$\varepsilon_{14}$						
		9	9	11	15						

*Proof.* This is because that  $T(G)\varepsilon_i \cong \mathcal{M}_{d_i}$  and that  $d_i^2 = \dim T(G)\varepsilon_i$  equals the number of linearly independent elements in the set  $\{x_j\varepsilon_i\}$  where  $x_j$  are the basis elements of  $T$ .  $\square$

Theorems 3.1 and 3.2 are combined as

$$165 = 1^2 + 2^2 + 3^2 + 3^2 + 5^2 + 6^2 + 9^2,$$

$$336 = 1^2 + 3^2 + 3^2 + 4^2 + 4^2 + 6^2 + 6^2 + 7^2 + 8^2 + 10^2,$$

$$758 = 1^2 + 1^2 + 1^2 + 3^2 + 3^2 + 4^2 + 6^2 + 7^2 + 8^2 + 8^2 + 9^2 + 9^2 + 11^2 + 15^2.$$

The degrees of irreducible complex representations afforded by every primitive central idempotents enable us to get the following structure theorem.

**Corollary 3.1.** *We have that*

$$T(PSL(2,7)) \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_5 \oplus \mathcal{M}_6 \oplus \mathcal{M}_9,$$

$$T(A_6) \cong \mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_4 \oplus \mathcal{M}_4 \oplus \mathcal{M}_6 \oplus \mathcal{M}_6 \oplus \mathcal{M}_7 \oplus \mathcal{M}_8 \oplus \mathcal{M}_{10},$$

$$T(S_6) \cong \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_4 \oplus \mathcal{M}_6 \oplus \mathcal{M}_7 \oplus \mathcal{M}_8 \oplus \mathcal{M}_8 \\ \oplus \mathcal{M}_9 \oplus \mathcal{M}_9 \oplus \mathcal{M}_{11} \oplus \mathcal{M}_{15}.$$

**Acknowledgment.** The second named author was supported by JSPS KAKENHI Grant Number JP25400014.

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